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Hyperkähler quotients and algebraic curves

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Abstract: We develop a graphical representation of polynomial invariants of unitary gauge groups, and use it to find the algebraic curve corresponding to a hyperkähler quotient of a linear space. We apply this method to four dimensional ALE spaces, and for the A_k, D_k, and E_6 cases, derive the explicit relation between the deformations of the curves away from the orbifold limit and the Fayet-Iliopoulos parameters in the corresponding quotient construction. We work out the orbifold limit of E_7, E_8, and some higher dimensional examples.

Keywords: Sigma Models, Superstring Vacua, Supersymmetric Effective Theories, Differential and Algebraic Geometry.

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Contents

1. Introduction

The two dual descriptions of D-branes as gravitational solitons of supergravity theories and as objects on which open strings can end make it possible to derive interesting new relations between gauge theories and gravity/string-theory. A celebrated example of this is the Maldacena conjecture [1]. To generalize the Maldacena conjecture to cases with more complicated gauge groups and matter content, as well as to cases with less supersymmetry, it is useful to study D-branes sitting on various spacetime singularities [2, 3, 4, 5].

This method was pioneered in [6], where D-branes on orbifold singularities of the type C^2/Z_k were studied. It was shown that the gauge theory realized on the brane is conveniently summarized by a "quiver diagram" from which one can read off the gauge group structure and the matter content. The orbifolds studied in that paper gave rise to quiver diagrams corresponding to the (extended) A_k Dynkin diagram; the generalization to the D and E series, corresponding to non-abelian orbifolds, was given in [7]. In this picture, the Fayet-Iliopoulos terms in the gauge theory are related to twisted sector moduli of the orbifold, so that turning them on corresponds to blowing up the orbifold singularity.

More complicated orbifolds, preserving less supersymmetry, were studied in, for example, [8, 9].

A complementary picture of the same models can be found using T-duality: the singularity generically transforms into a web of NS-branes [10] and the D-branes change their dimension giving configurations of the Hanany-Witten type [11]. In this dual picture, many features of the model become geometric. In particular, the Fayet-Iliopoulos terms are interpreted as various distances between branes. In many ways this dual picture is complementary to the original picture in that different things become easy to see whereas others, simple in the original picture, become harder to deal with in the T-dual version of the model. This T-duality has been studied in detail for the A-series [12] and the D-series [13, 14] but no duals have yet been found for the E-series.

One aim of this paper is to provide guidance to finding such a correspondence by constructing the map between the Fayet-Iliopoulos parameters, corresponding to the positions of various branes, and the deformations of the curve. In this paper, we find the algebraic curve corresponding to any manifold that is a hyperkähler quotient [15, 16] of a linear space. Such a quotient may be described in terms of a quiver diagram [6]. The cases when it yields a four-dimensional ALE manifold have been analyzed and have an A-D-E classifcation [17, 18]. In particular we include the Fayet-Iliopoulos parameters in the calculation of the curve for the A_k, D_k and E_6 cases. Remarkably we find that the curve in the E_6 case is identical to the Seiberg-Witten curve for certain $N = 2$ superconformal Yang-Mills theories with E_6 global symmetry and with the Fayet-Iliopoulos terms playing the roles of chiral superfield VEV's. It would be interesting if some duality or mirror symmetry were responsible for this apparent coincidence.

The paper is organized as follows. In section 2 we review the hyperkähler quotient in $N = 1$ superspace [15, 16]. In section 3, we describe the algebraic curves of these spaces, and derive them from the $N = 1$ superspace description of the hyperkähler quotient for the A_k [19], D_k [20], and E_k cases. In section 4, we discuss a number of issues and consider some higher dimensional examples outside the A-D-E classification.

2. Hyperkähler quotients

The hyperkähler quotient was introduced in [15] and was given a full mathematically rigorous presentation in [16]. It arises naturally when one gauges isometries of a nonlinear σ-model [21] in such a way as to preserve four dimensional $N = 2$ supersymmetry. In components ($N = 0$), it is the quotient of a constrained submanifold (the zero-set of the moment map [16]) by some real compact gauge group. In $N = 1$ superspace, the vector multiplet relaxes a part of the constraints, leaving only a holomorphic constraint, and enhances the gauge group to its complexification (subtleties pertaining to quotients by noncompact groups are discussed in [16, p. 548]).

Explicitly, we want to consider the hyperkähler quotient construction of 4-dimensional ALE spaces [18]. We start with a quaternionic vector space that we describe as an even dimensional complex space with n pairs of coordinates (z_+, z_-). In the language of supersymmetry, each pair of complex coordinates is called a hypermultiplet, and in $N = 1$ superspace, these are pairs of chiral superfields.

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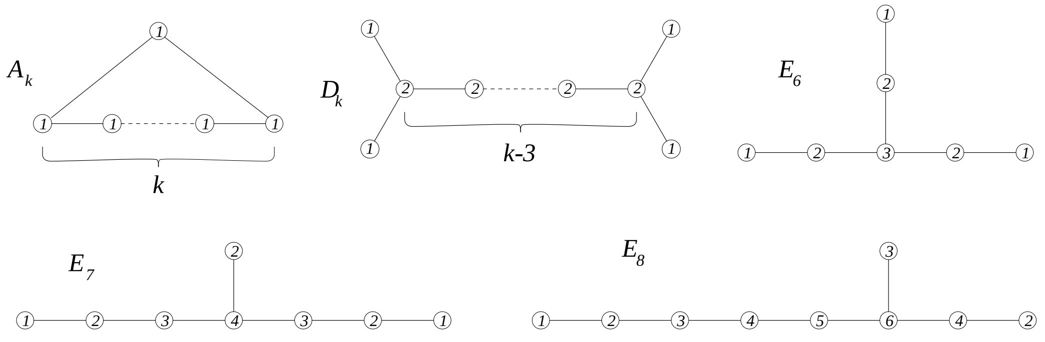

Figure 1: In the figure, each hyperkähler quotient with respect to a product gauge group $U(N_1) \times \cdots \times U(N_k)$ is represented as an extended Dynkin diagram for the A-D-E series of Lie groups; the i'th simple root, which has a label N_i in the Dynkin diagram, corresponds to a factor $U(N_i)$ in the gauge group and each link between two roots i, j corresponds to a hypermultiplet in the $(N_i, \bar{N}_j)$ representation.[1]

The Kähler potential of the metric is the superspace lagrangean [22]. The holomorphic moment map constraints take the form [15]

$$z_+ T_A z_- = \chi(T_A) \,, \tag{2.1}$$

where the T_A are generators of the gauge group (taken to be hermitean), and χ is an arbitrary character — i.e. a linear combination of the U(1) factors of the group:

$$\begin{aligned} z_+ T_A z_- &= 0 \,, \qquad A \notin \text{any } \mathrm{U}(1) \,, \\ &= b_A \,, \qquad A \in \text{any } \mathrm{U}(1). \end{aligned} \tag{2.2}$$

In superspace, these are called Fayet-Iliopoulos terms. The Kähler potential of the quotient space is found by solving a set of real equations for the $N = 1$ vector superfields V^A [15]:

$$z_+ e^{V^A T_A} T_A \bar{z}_+ - \bar{z}_- T_A e^{-V^A T_A} z_- = \hat{\chi}(T_A) \,, \tag{2.3}$$

and substituting the solution into the gauged flat space Kähler potential of the $z_\pm$'s [15]:

$$z_+ e^{V^A T_A} \bar{z}_+ + \bar{z}_- e^{-V^A T_A} z_- - V^A \hat{\chi}(T_A) \,, \tag{2.4}$$

where $\hat{\chi}$ is an independent character. The particular choices of gauge groups and representations are given in [18]; for a review see [19]. A summary is given in figure 1.

3. Algebraic curves

As stated above, four dimensional ALE hyperkähler manifolds are classified by the extended Dynkin diagrams corresponding to the A-D-E Lie groups [17]. As com-

[1] As the chiral superfields $z_\pm$ that make up the hypermultiplet are always in conjugate representations, the orientation of the links does not matter.

Classification	Polynomial	Deformations
A_k	$XY - Z^{k+1}$	$1, \ldots, Z^{k-1}$
D_k	$X^2 + Y^2 Z - Z^{k-1}$	$1, Y, Z, \ldots, Z^{k-2}$
E_6	$X^2 + Y^3 - Z^4$	$1, Y, Z, YZ, Z^2, YZ^2$
E_7	$X^2 + Y^3 + YZ^3$	$1, Y, Y^2, Z, YZ, Z^2, Y^2 Z$
E_8	$X^2 + Y^3 + Z^5$	$1, Y, Z, YZ, Z^2, Z^3, YZ^2, YZ^3$

Table 1: The ADE series.

plex manifolds, they can be described by holomorphic curves[2] in $\mathbb{C}^3$ with coordinates X, Y, Z. The curves have a leading piece corresponding to the orbifold limit of the spaces, and deformation parameters corresponding to the character (Fayet-Iliopoulos) terms of the quotient construction. The curves are summarized in table 1.

The curve corresponding to a given hyperkähler quotient can be constructed by finding all the (gauge group) invariant holomorphic polynomials modulo the holomorphic constraints (2.2); for the A-D-E spaces, we find exactly three polynomials that satisfy an algebraic constraint, which, after suitable redefinitions is precisely the equation of the corresponding algebraic curve. The construction is in the spirit of Seiberg's construction of effective superpotentials [23].

We now describe the actual calculations. To find the correct variables and derive the curve it is useful to employ the following graphic "bug calculus".

As described above, we can represent any hyperkähler quotient by a product gauge group $\mathrm{U}(N_1) \times \cdots \times \mathrm{U}(N_k)$ as a quiver diagram,[3] where the i'th node, labeled N_i, corresponds to a $\mathrm{U}(N_i)$ gauge group and each link (including orientation) between two nodes i, j corresponds to a hypermultiplet in the $(N_i, \bar{N}_j)$ representation. It is therefore natural to represent any invariant that can be obtained by multiplying the chiral fields of the model as a closed oriented loop in the quiver diagram.

The holomorphic constraints (2.2) can be represented in bug calculus. Each gauge group (i.e. node) has its own constraint. For an "endpoint" the constraint is shown in figure $2a$ and for a point in a chain the constraint is shown in figure $2b$. For more complicated junctions one generalizes this keeping in mind that the sign of each term is determined by the orientation of the link it sits on. This is shown for two of the junctions that appear in this paper: the three and four-point vertices in figures $2c$ and $2d$.

A second important ingredient is the so-called Schouten identity. This allows us to untwist various loops showing that complicated variables can be written as products of simpler ones. They are derived by observing that the totally antisymmetric

[2]We should, strictly speaking, use the term *algebraic variety* instead of algebraic curve since the latter term is reserved for one-complex dimensional objects and we are discussing higher dimensional objects.

[3]A quiver diagram is essentially a Dynkin diagram with arrows on the links indicating an orientation; when we construct invariant polynomials, we need to keep track of the orientation.

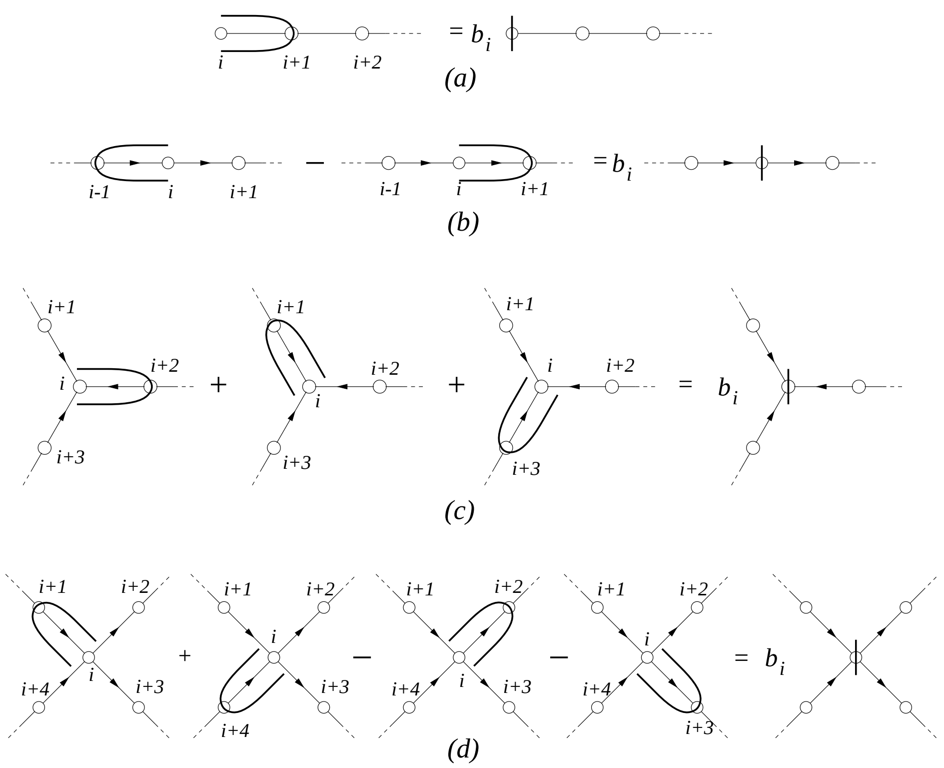

Figure 2: The bug calculus. b_i is the Fayet-Iliopoulos parameter associated to the i'th node, and a vertical bar through the i'th node represents a $\mathrm{U}(N_i)$ Kronecker-δ.

product of $n+1$ n-dimensional indices is identically zero. The simplest identity holds for one dimensional matrices and simply says that one dimensional matrices commute. Graphically this means that on any node representing a U(1) gauge group we are allowed to split the lines and reconnect them in any way as long as we respect the orientation of the loops.

The Schouten identity for two dimensional matrices looks slightly more complicated. It can be derived from

$$M_{k_1}^{[i_1}N_{k_2}^{i_2}K_{k_3}^{i_3]} = 0\,. \tag{3.1}$$

If we contract the indices we can derive the following identity appropriate for our purposes

$$\begin{aligned}\mathrm{Tr}(\{M,N\}K) = {}& \mathrm{Tr}(MN)\,\mathrm{Tr}(K) + \mathrm{Tr}(MK)\,\mathrm{Tr}(N) + \\ & + \mathrm{Tr}(NK)\,\mathrm{Tr}(M) - \mathrm{Tr}(M)\,\mathrm{Tr}(N)\,\mathrm{Tr}(K)\,. \end{aligned}\tag{3.2}$$

In principle we could also implement this identity graphically. However, in practice it is easier to use it in algebraic form and then to go on and use the graphic methods on each term separately.

These are all the tools we need to derive the algebraic curve for any hyperkähler quotient corresponding to an arbitrary quiver diagram: we draw closed loops of

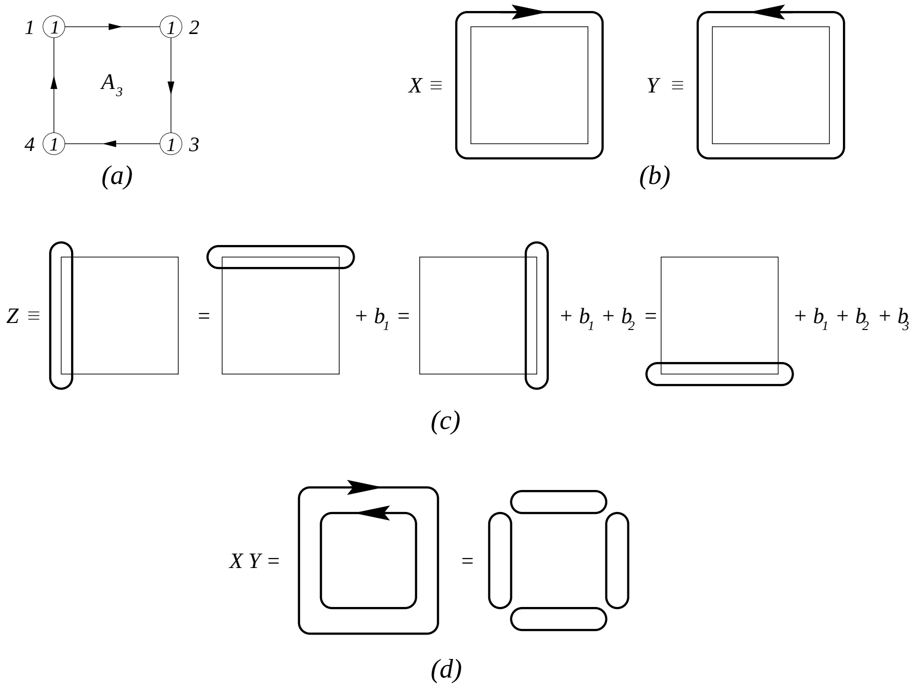

Figure 3: Diagramatic representation of the A_3 invariants, moment map constraints, and algebraic curve.

increasing order in the number of links, and use the bug calculus to find the independent nonvanishing invariants. In practice, we first consider the orbifold limit, as then the relations between the invariants are simpler; the final calculations away from this limit then follow precisely the same route, but yield many more terms. The independent invariants are good coordinates on the moduli space. When we find no new independent invariants, we have all the coordinates of the moduli space. To find the algebraic curve, we consider the product of the highest order invariant with its orientation reversed image and use Schouten identities to express it as a product of lower invariants; for the D_k and E_k but not the A_k cases, the orientation reversed loop is proportional to the original invariant plus algebraic functions of the lower invariants. For the A_k and D_k cases, the U(1) Schouten identity is all we need.

We illustrate the method with two examples: A_3 and D_4, and then describe our results for the general case. Figure 3 describes the full calculation for A_3: $3a$ shows the quiver diagram, $3b$ defines two of the independent variables, X, Y, $3c$ defines the variable Z and uses the relation $2b$ to express other similar diagrams in terms of it (note that the Fayet-Iliopoulos terms satisfy $\sum_1^4 b_i = 0$), and finally, using the relation in $3c$, $3d$ gives the algebraic curve in diagramatic form.[4]

$$XY = Z(Z - b_1)(Z - b_1 - b_2)(Z - b_1 - b_2 - b_3)\,. \tag{3.3}$$

[4]This can be made to match the curve given in table 1 by shifting $Z \to Z + \frac{1}{4}(3b_1 + 2b_2 + b_3)$.

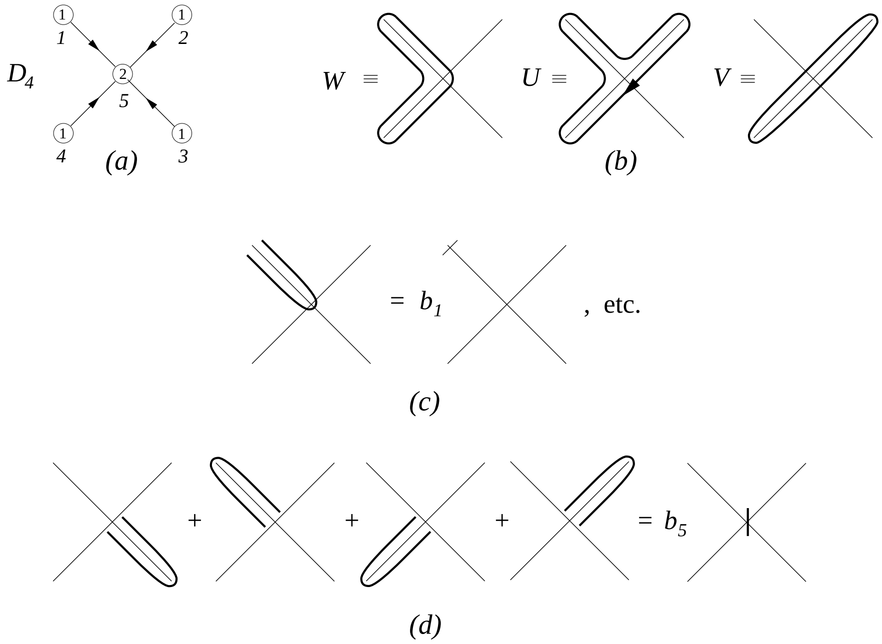

Figure 4: Diagramatic representation of the D_4 invariants and moment map constraints.

The calculation for the general A_k is completely analogous, and gives the curve $XY = \prod_{i=0}^{k}(Z - \sum_{j=1}^{i} b_j)$, where again $\sum_1^{k+1} b_i = 0$.

Figures 4 and 5 describe the calculation for D_4. Figure 4a shows the quiver diagram, figure 4b defines the three independent variables and figures 4c and 4d give the constraints 2c and 2d for this particular case; as for the A_k case, the Fayet-Iliopoulos coefficients are constrained: $\sum_1^4 b_i = 2b_5$.

Figure 5a expresses a four-link diagram in terms of the basic four-link diagrams W and V. Figures 5b and 5c relate U to its orientation reversed image. Figure 5d yields the algebraic curve in diagramatic form. Substituting 5a–c and similar relations into 5d we find

$$U^2 + U[(b_4 - b_1)V + (b_4 - b_2)W + a_1] - W^2 V - WV^2 + a_2 WV = 0\,, \tag{3.4}$$

where

$$a_1 \equiv b_4(b_5 - b_4)(b_5 - b_4 - b_3)\,, \qquad a_2 \equiv \frac{1}{2}\left[\sum_{i\neq 3}^{4} b_i(b_5 - b_i) - b_3(b_5 - b_3)\right]. \tag{3.5}$$

Making the following redefinitions,

$$\begin{aligned} U &= \frac{1}{2}[X + (b_1 - b_4)V + (b_2 - b_4)W - a_1]\,, \\ V &= \frac{1}{2}\left[Y - W + a_2 - \frac{1}{2}(b_1 - b_4)\left(b_2 - \frac{1}{2}(b_1 + b_4)\right)\right], \\ W &= -Z - \frac{1}{4}(b_1 - b_4)^2\,, \end{aligned} \tag{3.6}$$

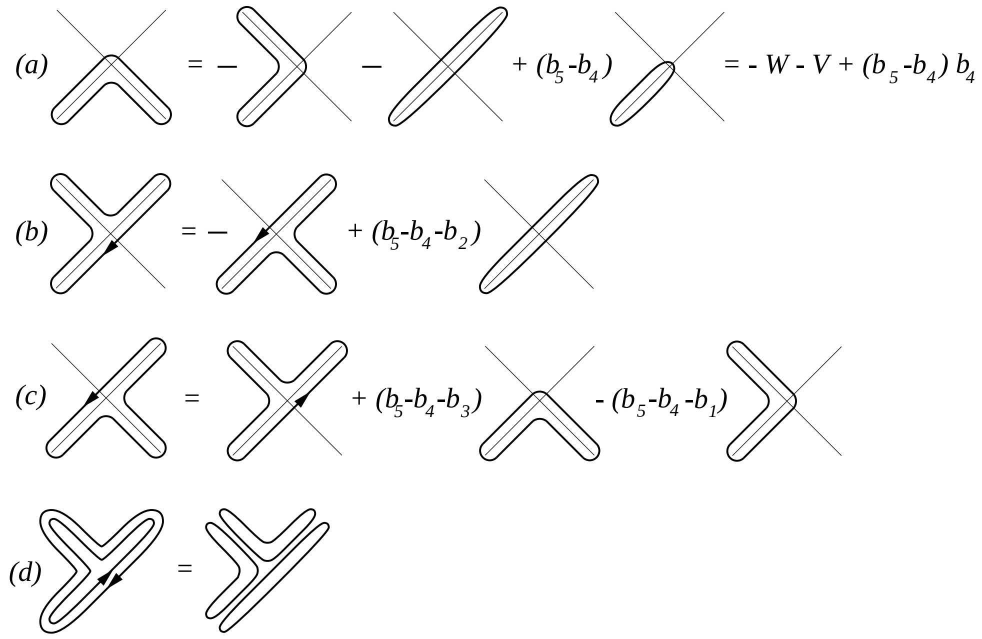

Figure 5: Some typcial calculations for the D_4 example.

we find a standard form of the algebraic curve for D_4:

$$X^2 + Y^2 Z - Z^3 + \alpha_0 Y - \sum_1^3 \alpha_i Z^{i-1} = 0\,. \tag{3.7}$$

The coefficients α_i are expressed in terms of the Fayet-Iliopoulos parameters b_i as follows:

$$\begin{aligned}
\alpha_0 &= \frac{1}{8}(b_1^2 - b_4^2)(b_2^2 - b_3^2)\,,\\
\alpha_1 &= \frac{1}{32}\left[(b_1^2 + b_4^2)(b_2^2 - b_3^2)^2 + (b_2^2 + b_3^2)(b_1^2 - b_4^2)^2\right],\\
\alpha_2 &= \frac{1}{16}\left[(b_1^2 - b_4^2)^2 + (b_2^2 - b_3^2)^2 + 4(b_1^2 + b_4^2)(b_2^2 + b_3^2)\right],\\
\alpha_3 &= \frac{1}{2}\sum_1^4 b_i^2\,.
\end{aligned} \tag{3.8}$$

For the general D_k, we label the nodes as indicated in figure $6a$; the basic variables U, V, W are defined by analogy to the D_4 case (figure $4b$, and are shown in figure $6b$. The Fayet-Iliopoulos coefficients b_i (associated to the i'th node) satisfy the constraint

$$\sum_1^4 b_i = 2\sum_5^{k+1} b_i\,. \tag{3.9}$$

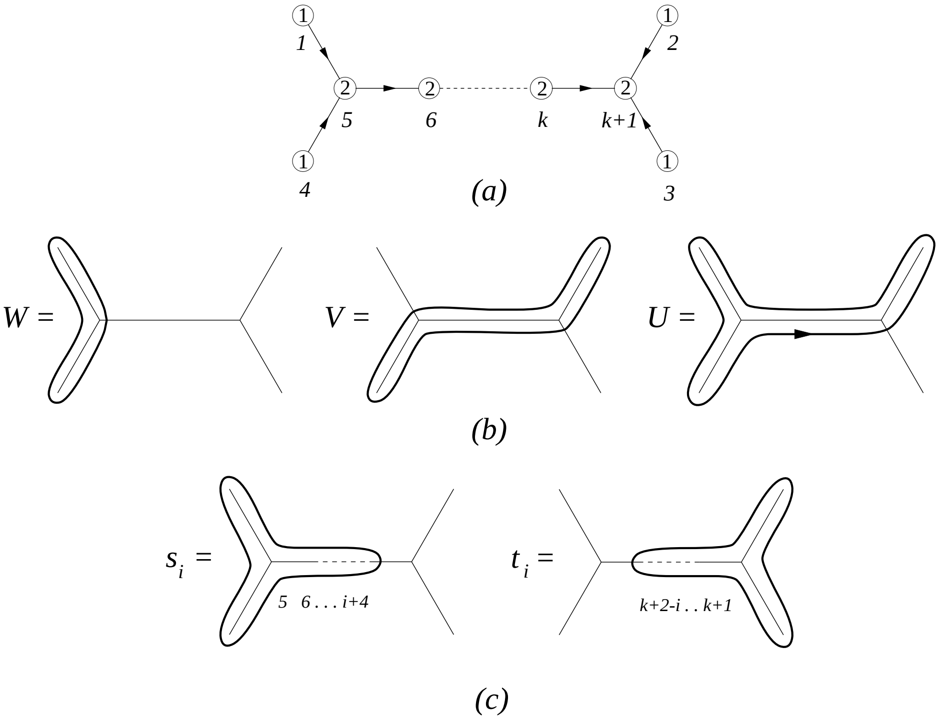

Figure 6: The basic invariants for D_k and some other useful invariant quantities.

We express the curve in terms of certain polynomials in W that we define recursively as follows:

$$
\begin{aligned}
s_i &= [b_1 + b_4 - b_{i+3} - 2\beta_{i-2}]\, s_{i-1} + [s_1 - (b_1 - \beta_{i-2})(b_4 - \beta_{i-2})]\, s_{i-2}\,, \\
t_i &= [b_1 + b_4 + b_{k+3-i} - 2\beta_{k-1-i}]\, t_{i-1} + [s_1 - (b_1 - \beta_{k-1-i})(b_4 - \beta_{k-1-i})]\, t_{i-2}\,, \\
\beta_i &\equiv \sum_{j=5}^{i+4} b_j\,;
\end{aligned}
\tag{3.10}
$$

their graphical expression is given in figure 6c). The initial conditions are

$$
s_0 = t_0 = 0\,, \qquad s_1 = W\,, \qquad t_1 = W + \beta_{k-3}(\beta_{k-3} - b_1 - b_4) + \frac{1}{2}(b_1^2 + b_4^2 - b_2^2 - b_3^2)\,. \tag{3.11}
$$

For arbitrary k, the curve in the form analogous to (3.4) is

$$
U^2 - WV^2 + U[P(W) + (b_4 - b_1)V] + VQ(W) = 0\,, \tag{3.12}
$$

where P and Q are polynomials of order $\mathcal{O}(W^{[\frac{k-3}{2}]+1})$ and $\mathcal{O}(W^{[\frac{k-2}{2}]+1})$, respectively:

$$
\begin{aligned}
P &= \frac{b_3 - b_2}{t_1}(t_{k-1} - b_1 t_{k-2}) + s_{k-2} - \frac{b_1 + b_3 - \beta_{k-3}}{s_1}[s_{k-1} - (b_1 - \beta_{k-3})s_{k-2}]\,, \\
Q &= -\frac{s_1}{t_1}[t_{k-1} + (b_3 - \beta_{k-3})t_{k-2}]\,.
\end{aligned}
\tag{3.13}
$$

Though they do not have a graphical representation, s_i, t_i for $i > k-3$ are defined by the recursion relations (3.10); we also need $b_i = 0$ for $i > k+1$, $\beta_i = 0$ for $i < 1$, and we take $b_{k+3-i} = 0$, and not b_4, for $i = k-1$. Because of the initial conditions (3.11), s_i/s_1 and t_i/t_1 are polynomials in W.

The curve (3.12) can be put into the standard form by redefinitions analogous to (3.6):

$$\begin{aligned} U &= \frac{1}{2}[X + (b_1 - b_4)V - P(W)]\,, \\ V &= \frac{1}{2}\left[Y - \frac{R(Z) - R(0)}{Z}\right], \\ W &= -Z - \frac{1}{4}(b_1 - b_4)^2\,, \\ R(Z) &\equiv \left[Q(W(Z)) + \frac{1}{2}(b_1 - b_4)P(W(Z))\right]_{W(Z) = -Z - \frac{1}{4}(b_1 - b_4)^2}\,, \end{aligned} \tag{3.14}$$

which gives

$$X^2 + Y^2 Z + 2R(0)Y - \left[\frac{R^2(Z) - R^2(0)}{Z} + P^2(W(Z))\right] = 0\,. \tag{3.15}$$

Calculating the first six examples, we are able to rewrite the curve (3.15) explicitly in terms of the Fayet-Iliopoulos coefficients:

$$X^2 + Y^2 Z - 2Y \prod_1^k B_i - \frac{\prod_1^k (Z + B_i^2) - \prod_1^k B_i^2}{Z} = 0\,, \tag{3.16}$$

where

$$\{B_i\} \equiv \left\{\frac{1}{2}(b_1 - b_4), \frac{1}{2}(b_2 - b_3), \frac{1}{2}(b_1 + b_4), \frac{1}{2}(b_1 + b_4) - b_5, \ldots, \frac{1}{2}(b_1 + b_4) - \sum_5^{k+1} b_i\right\}. \tag{3.17}$$

In the orbifold limit, $B_i = 0$ which agrees with the entry for D_k in table 1. After completing our calculation, we realized that the same expression for the deformation in terms of the Fayet-Iliopoulos parameters had been deduced by completely different methods in [20].

We note that the quantities that enter in both the A_k and the D_k cases are related to the weights of the fundamental representation of the Lie algebra in question. If we think of each Fayet-Iliopoulos parameter as the simple root associated to its node in the Dynkin diagram, then the expressions that occur ($\sum_{j=1}^i b_j$ in the A_k case and the B_i (3.17) in the D_k case) are the weights of the fundamental representation. More precisely, since the Fayet-Iliopoulos terms are scalars whose value may be freely chosen, the quantities entering the curve should be associated with $v \cdot \lambda$ where λ

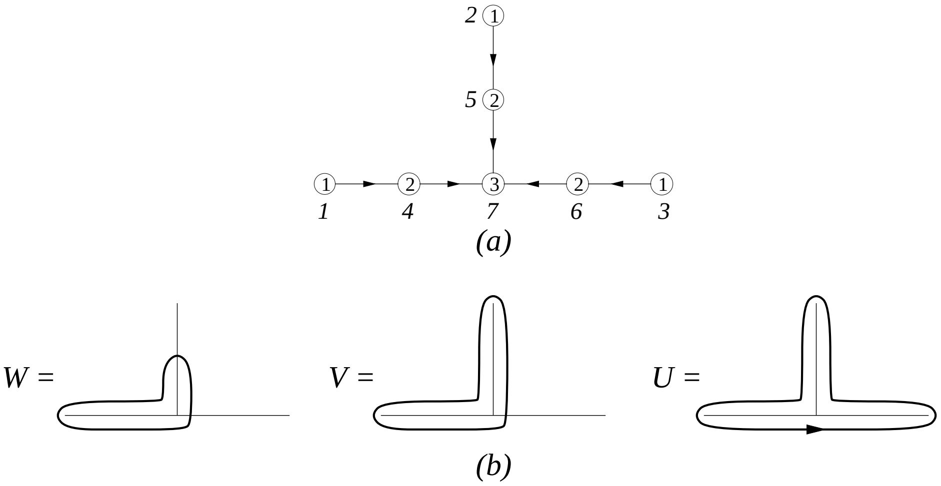

Figure 7: The E_6 invariants.

is the particular weight and v is a vector of the same dimension as the rank of the group. This ensures that we can choose the Fayet-Iliopoulos parameters to be zero, corresponding to a zero value for v and when we turn on the Fayet-Iliopoulos parameters it corresponds to giving v a non-zero value such that the quantities above agree. This observation will be used later to write the result for the E_6 curve in a nice form.

We now turn to the E-series. The labeling of the nodes for E_6 is given in figure $7a$, and the invariant polynomials U, V and W are defined in figure $7b$. The relation between U and its orientation reversed image $\bar{U}$ is:

$$U + \bar{U} = -W^2 + A_W W + A_V V + A_0 \,, \tag{3.18}$$

where

$$\begin{aligned}
A_V &\equiv \sum_1^3 b_i(b_i - b_{i+3}) - \frac{1}{4}\left(\sum_1^3 b_i + b_7\right)^2 , \\
A_W &\equiv \frac{1}{4}\left(\sum_1^3 b_i + b_7\right)(b_4 + b_5 - b_1 - b_2 + b_7)(b_1 + b_2 - b_3 + b_7) - \\
&\quad - b_1 b_5(b_1 - b_4) + b_2(2b_1 - b_4)(b_2 - b_5) \\
A_0 &\equiv -b_2(b_1 + b_3 - b_4 - b_6 - b_7)C_0 \,, \\
C_0 &\equiv b_1(b_1 - b_4)(b_1 - b_4 - b_7)(b_1 - b_4 - b_6 - b_7) \,.
\end{aligned} \tag{3.19}$$

Just as in the D_4 case (cf. figure $5d$) the curve follows from expressing U^2 as $\mathrm{U}(-\bar{U} + \cdots)$. The result is

$$U^2 - \mathrm{U}(-W^2 + A_W W + A_V V + A_0) + V\left[V + C_W W + C_0\right]\left[V + D_W W + D_0\right] = 0 \tag{3.20}$$

where

$$
\begin{aligned}
C_W &\equiv b_1 - b_2 - b_3 + b_5 + b_6 + b_7\,, \\
D_W &\equiv b_1 - b_2 + b_3 - b_4 - b_6 - b_7\,, \\
D_0 &\equiv \frac{1}{27}\Bigg[2b_1^3 - \frac{27}{8}\left(\frac{1}{3}b_1 - b_2 + b_3 - b_7\right)\left(\frac{1}{3}b_1 + b_2 - b_3 - b_7\right) \times \\
&\qquad \times \left(\frac{1}{3}b_1 - b_2 + b_3 - 2b_6 - b_7\right) + \\
&\qquad + 3b_1\Big\{(2b_1 - b_2 - 2b_4 + b_5)(b_1 - 2b_2 - b_4 + 2b_5) - \\
&\qquad\qquad - 2\left(b_6^2 + (b_2 - b_3)(b_2 - b_3 + b_6)\right)\Big\}\Bigg] \times \\
&\quad \times (b_1 + b_3 - b_4 - b_6 - b_7)\,.
\end{aligned} \tag{3.21}
$$

Performing the following shifts

$$
\begin{aligned}
U &= X + \frac{1}{2}(-W^2 + A_W W + A_V V + A_0)\,, \\
V &= Y - \frac{1}{3}\left[C_0 + D_0 + (C_W + D_W)W - \frac{A_V^2}{4}\right], \\
W &= \sqrt{2}Z + \frac{1}{6}\left[3A_W - \left(A_V + \frac{2}{3}C_W D_W\right)(C_W + D_W) + \frac{4}{9}\left(C_W^3 + D_W^3\right)\right],
\end{aligned} \tag{3.22}
$$

the curve (3.20) is brought to the standard form

$$
X^2 + Y^3 - Z^4 + P(Z) + Q(Z)Y = 0\,, \tag{3.23}
$$

where the polynomials $P(Z)$ and $Q(Z)$ are second order in Z. The coefficients in terms of the Fayet-Iliopoulos parameters may be found by substituting (3.18), (3.19), (3.21), and (3.23) into (3.20). Direct evaluation leads to a horrible mess, but the polynomials may be expressed in terms of Casimir operators of E_6; remarkably, when we do this, we find the algebraic curve given in [24]. We now present the details of this description.

The Casimirs can be defined as the coefficients of the polynomial

$$
\det(x - \Phi)\,, \tag{3.24}
$$

where Φ is a matrix in the fundamental representation of E_6. We can always rotate Φ into some element in the Cartan subalgebra $v \cdot H$ where v is an arbitrary six dimensional vector. An explicit representation for the matrices H can be found in terms of the weights λ of the fundamental representation, since the weight vectors can be thought of as normalized eigenvectors of the Cartan operators with the weights

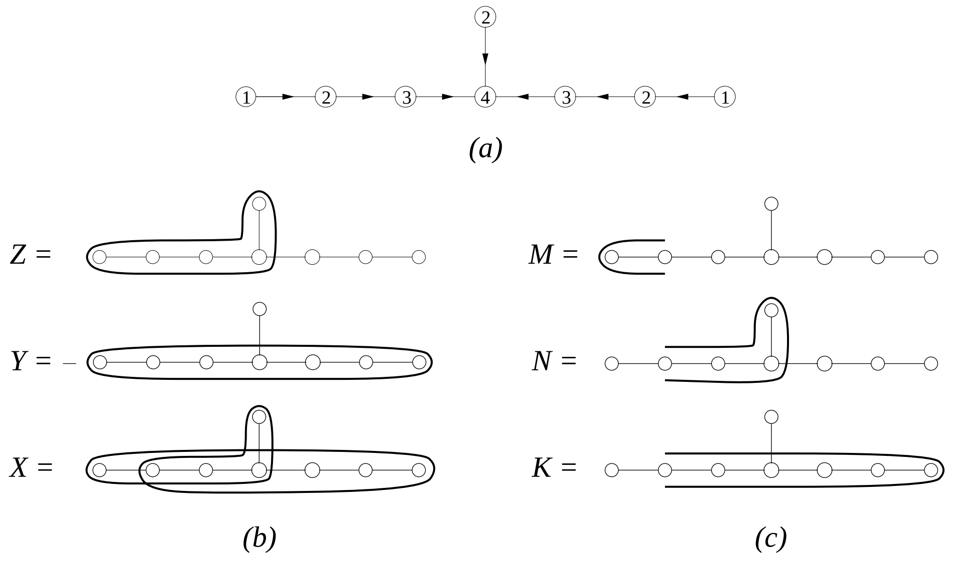

Figure 8: The E_7 invariants and some useful matrices.

as eigenvalues. The Cartan operators are thus represented as diagonal matrices with the particular weights on the diagonal, and we have

$$\Phi = v \cdot H = \begin{pmatrix} v \cdot \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & v \cdot \lambda_{27} \end{pmatrix}. \tag{3.25}$$

The terms on the diagonal are just the expressions for the weights in terms of the Fayet-Iliopoulos parameters as discussed at the end of the derivation of the curve for the D_k case. Thus we have found a way to express the Casimirs in terms of the Fayet-Iliopoulos parameters. More details as well as the final result for the curve can be found in the appendix.

It is natural to conjecture that the relation between the deformation parameters of the curve and the Fayet-Iliopoulos parameters follows the same pattern for the higher exceptional algebras [25, 26]. The Fayet-Iliopoulos are to be thought of as the simple roots of the algebra and the Casimirs of the fundamental representation of the algebra (expressed in terms of the simple roots and thus in terms of the Fayet-Iliopoulos paramters) give the deformation parameters of the curve.

We note that this explicit expression stands in contrast to the implicit one of [20], which involves inverting elliptic integrals.

We now turn to the E_7 and E_8 cases.

For E_7 and E_8, we consider only the orbifold limit (no Fayet-Iliopoulos terms). In figure 8, the quiver diagram (figure 8a and the basic invariant polynomials (figure 8b for E_7 are shown. We have verified that all other possible invariants either vanish or are polynomials in these basic variables. In the E_7 case the orientation reversal of the highest dimension graph X is just $-X$, but when we multiply them together

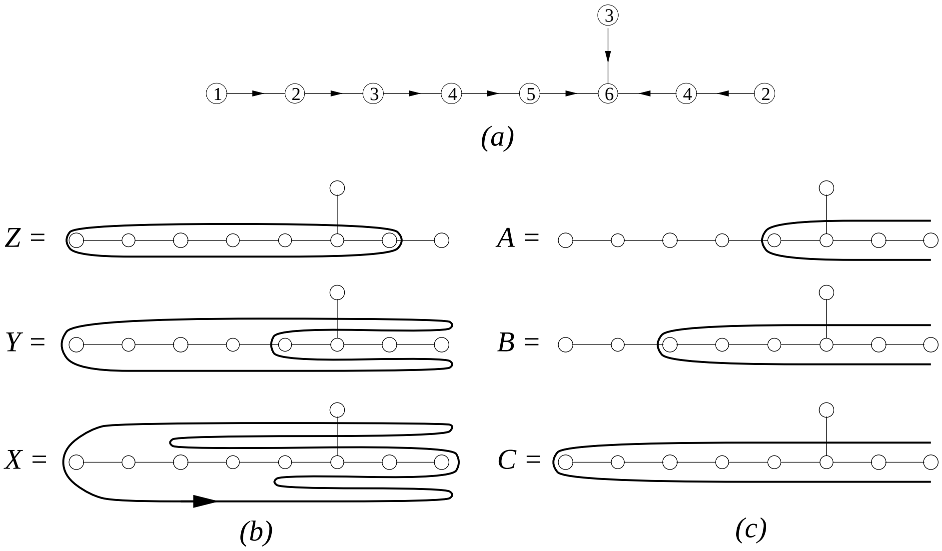

Figure 9: The E_8 invariants and some useful matrices.

the result does not immediately factorize into a sum over products of the basic lower dimensional variables. It therefore turns out to be convenient do define the traceless 2×2 matrices M, N and K as in figure 7c. Using the bug calculus it is possible to derive the following useful relations

$$\begin{aligned} \mathrm{Tr}\,(NK) &= -Z^2\,, \\ \mathrm{Tr}\,(MK) &= -Y\,, \\ \mathrm{Tr}\,(MN) &= Z\,, \\ \mathrm{Tr}\left(N^2\right) &= -2Y\,. \end{aligned} \tag{3.26}$$

Using these matrices we can write the square of the highest dimensional invariant as $X^2 = Y\,\mathrm{Tr}\,(MNKN)$. To be able to use the Schouten identity (3.2) we rewrite the trace in terms of anticommutators by anticommuting the leftmost matrix all the way to the right. We can now rewrite the trace in terms of products of traces of fewer matrices. The result, dropping terms that vanish, is

$$\mathrm{Tr}\,(MNKN) = \mathrm{Tr}\,(MN)\,\mathrm{Tr}\,(KN) - \frac{1}{2}\,\mathrm{Tr}\left(N^2\right)\mathrm{Tr}\,(MK)\,, \tag{3.27}$$

which, using the relations (3.26) gives the curve

$$X^2 + Y^3 + YZ^3 = 0\,. \tag{3.28}$$

Finally we turn to the E_8 ALE space; the quotient gauge group and matter content are described by the quiver diagram in figure 9a, and the basic invariant polynomials are defined in figure 9b. Again, we have verified that all other possible invariants either vanish or are polynomials in these basic variables. Since there is only one U(1) group in this case it is not possible to simply factorize the square of the highest dimensional invariant X into a product of lower dimensional ones and we must again use the two dimensional Schouten identity (3.2). Therefore it is useful to define the traceless 2×2 matrices A, B and C in figure 9c. Using the bug calculus we derive the following identities

$$\begin{aligned} \mathrm{Tr}\,(ABC) &= X\,, \\ \mathrm{Tr}\,(AB) &= 0\,, \\ \mathrm{Tr}\,(BC) &= Z^2\,, \\ \mathrm{Tr}\left(A^2\right) &= -2Z\,, \\ \mathrm{Tr}\left(B^2\right) &= -2Y\,. \end{aligned} \tag{3.29}$$

Squaring the highest dimensional invariant X and using the one dimensional Schouten identity we can write the result as $X^2 = \mathrm{Tr}\,(ABCABC)$. Rewriting the trace in terms of anticommutators by using the same trick as in the E_7 case we get

$$X^2 = \mathrm{Tr}\,(AC)\,\mathrm{Tr}\left(AB^2C\right) - \mathrm{Tr}\,(BC)\,\mathrm{Tr}\,(ACAB)\;, \tag{3.30}$$

and using the same trick once again on the traces with four matrices we find

$$\begin{aligned} \mathrm{Tr}\left(AB^2C\right) &= \frac{1}{2}\,\mathrm{Tr}\,(AC)\,\mathrm{Tr}\left(B^2\right), \\ \mathrm{Tr}\,(ACAB) &= -\frac{1}{2}\,\mathrm{Tr}\left(A^2\right)\mathrm{Tr}\,(BC)\,. \end{aligned} \tag{3.31}$$

Finally, using (3.29), we arrive at the following result for the curve

$$X^2 + Y^3 + Z^5 = 0\,. \tag{3.32}$$

4. Other examples

There is something a bit surprising about our calculations: aside from those few nodes where we used the Schouten identities, our calculations did not in any way refer to the gauge group associated with each node of the quiver. Thus if we change the Dynkin indices of those nodes where we did not use a Schouten identity, we get the same invariants and the same algebraic curve. However, when we consider the hyperkähler quotient, this is clearly nonsensical: the delicate balance between the dimension of the gauge group and the number of hypermultiplets is achieved only for the correct Dynkin indices: e.g., for D_4, if the central node is changed from U(2) to U(n), the resulting hyperkähler quotient has zero or negative dimension. The resolution of this paradox becomes clear when we express the fields of the hypermultiplets in "spherical"-type coordinates, that is, in terms of goldstone modes that transform under the gauge group ("angles"), and the invariants ("radii"). When the

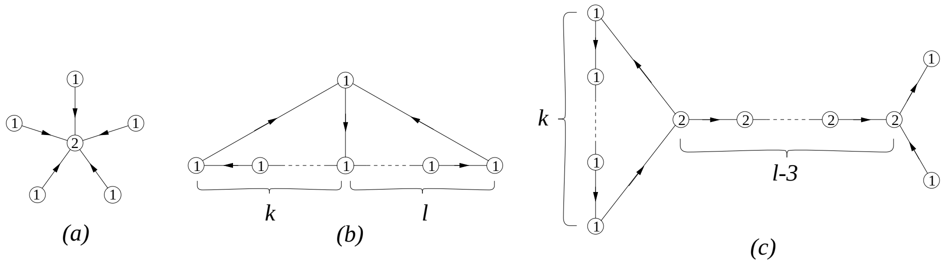

Figure 10: Some higher dimensional examples.

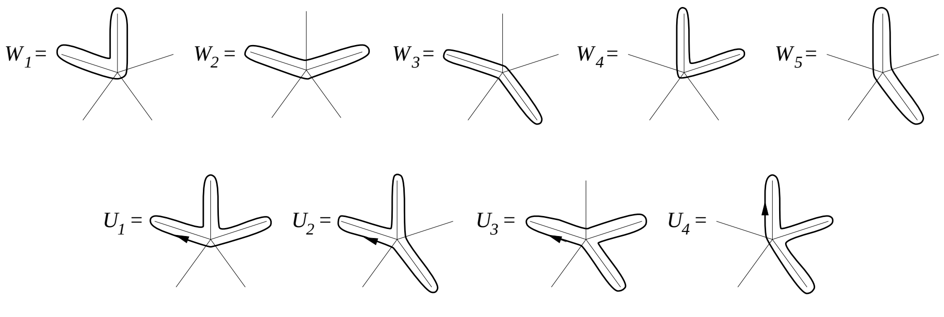

Figure 11: The invariants for an eight-dimensional analog of D_4.

dimension of the hyperkähler quotient is zero, the hypermultiplet action depends only on the goldstone modes, and the invariants that live on the algebraic curve do not enter the dynamics (one could imagine that under some circumstances these invariants correspond to dynamically generated states of the theory, and then the nontrivial hyperkähler quotient manifold would arise); if the hyperkähler quotient would give rise to a negative dimension space, then the hypermultiplet action is not only independent of the invariants, but even of some of the goldstone modes.

The graphical methods that we have developed can be used to find the algebraic curves for higher dimensional ALE spaces. A few typical examples are shown in figure 10. We have analyzed only the orbifold limits of these examples.

For figure 10a, there are nine linearly independent invariants as defined in figure 11. Direct application of our method gives ten polynomial equations that these invariants satisfy. However, a little calculation shows that these ten equations are generated by five equations, leaving a complex four dimensional space as expected from the hyperkähler quotient:

$$\begin{gathered} U_1^2 = -W_1 W_2 W_4 \,, \qquad U_2^2 = -W_2 W_3 W_5 \,, \\ U_3^2 = -W_2 W_3 \sum_1^5 W_i \,, \qquad U_4^2 = -W_4 W_5 \sum_1^5 W_i \,, \\ 4 \prod_2^5 W_i = \left(W_2 W_5 + W_3 W_4 + W_1 \sum_1^5 W_i \right)^2 . \end{gathered} \tag{4.1}$$

This space should be an interesting nontrivial extension of D_4 to higher dimensions.

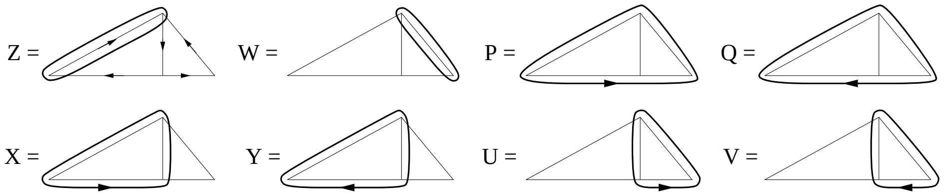

Figure 12: The invariants for eight-dimensional Calabi ALE spaces.

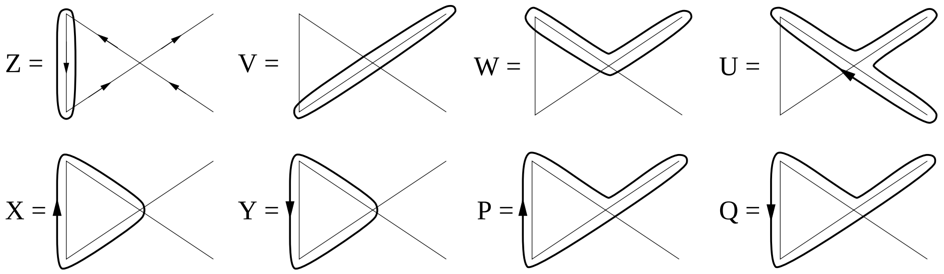

Figure 13: The invariants for another eight-dimensional example.

It is straightforward to find the ten equations with the Fayet-Iliopoulos terms turned on; however, in that case, the reduction to five equations seems to be tedious.

For figure 10b, there are eight linearly independent invariants as defined in figure 12. These obey four relations, leaving a complex four dimensional space:

$$\begin{aligned} XY &= Z^k(Z+W)\,, \qquad & UV &= W^l(Z+W)\,, \\ XU &= P(Z+W)\,, \qquad & YV &= Q(Z+W)\,. \end{aligned} \tag{4.2}$$

Note that in the orbifold limit, away from the subspace $Z+W=0$, this is just the product space $A_k \times A_l$. These spaces are well understood higher dimensional analogs of the A_k ALE spaces; examples were constructed as hyperkähler quotients in [15], though they had been proposed earlier as hyperkähler spaces in [27].

For figure 10c, there are eight linearly independent invariants as defined in figure 13. These obey four relations, leaving a complex four dimensional space:

$$\begin{aligned} XQ &= Z(ZV+U)\,, \qquad & YP &= Z(ZW-U)\,, \\ XY &= Z(Z^2+W-V)\,, \qquad & PQ &= ZWV\,. \end{aligned} \tag{4.3}$$

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A. Details on E_6

The algebraic curve for the E_6 case including the Fayet-Illiopoulos parameters is

$$
\begin{aligned}
& X^2+Y^3+\left[\frac{1}{3}P_2Z^2-\frac{2}{3}P_5Z+\frac{8}{15}P_8-\frac{11}{45}P_2P_6+\frac{7}{432}P_2^4\right]Y+ \\
& +\Bigg[-Z^4-\left(\frac{2}{3}P_6-\frac{7}{108}P_2^3\right)Z^2+\left(\frac{8}{21}P_9-\frac{1}{18}P_2^2P_5\right)Z-\frac{32}{135}P_{12}+ \\
& +\frac{298}{18225}P_2^2P_8+\frac{101}{218700}P_2^3P_6-\frac{13}{405}P_6^2+\frac{49}{1049760}P_2^6+\frac{19}{3645}P_2P_5^2\Bigg]=0\,, \qquad \text{(A.1)}
\end{aligned}
$$

where the P_i is the Casimir of the i'th order. It can be found as the coefficient of x^{27-i} term of the polynomial

$$\det\left(x-v\cdot H\right) \qquad \text{(A.2)}$$

where $v\cdot H$ is given in terms of the weights λ of the fundamental representation as $v\cdot H=\mathrm{diag}\left(v\cdot\lambda_1\cdots v\cdot\lambda_{27}\right)$. In particular, if we define $\chi_n\equiv\mathrm{Tr}\left[(v\cdot H)^n\right]$, we find that we can write the relevant P_i as follows

$$
\begin{aligned}
P_2 &= -\frac{1}{2}\chi_2\,, \\
P_5 &= \frac{1}{5}\chi_5\,, \\
P_6 &= -\frac{1}{6}\chi_6-\frac{1}{96}\chi_2^3\,, \\
P_8 &= -\frac{1}{8}\chi_8+\frac{1}{12}\chi_2\chi_6+\frac{1}{4608}\chi_2^4\,, \\
P_9 &= \frac{1}{9}\chi_9-\frac{1}{14}\chi_2\chi_7+\frac{1}{48}\chi_2^2\chi_5\,, \\
P_{12} &= -\frac{1}{12}\chi_{12}+\frac{11}{1920}\chi_2^2\chi_8+\frac{1}{72}\chi_6^2-\frac{1}{1440}\chi_2^3\chi_6+\frac{17}{2400}\chi_2\chi_5^2+\frac{1}{663552}\chi_2^6\,, \qquad \text{(A.3)}
\end{aligned}
$$

where, for E_6[5],

$$
\begin{aligned}
\chi_4 &= \frac{1}{12}\chi_2^2\,, \\
\chi_7 &= \frac{7}{24}\chi_2\chi_5\,, \\
\chi_{10} &= \frac{7}{40}\chi_5^2+\frac{3}{8}\chi_2\chi_8-\frac{7}{144}\chi_2^2\chi_6+\frac{7}{41472}\chi_2^5\,. \qquad \text{(A.4)}
\end{aligned}
$$

[5] After we completed our calculations, we were informed that such formulas are derived in great generality in [28].

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To express the Casimirs in terms of the Fayet-Iliopoulos parameters, we write the weights of the fundamental representation in terms of the simple roots and recall that each b_i can be thought of as the scalar product between the v and its corresponding simple root.[6] Doing this we find

$$
\begin{aligned}
v\cdot\lambda_1 &= \frac{1}{3}\,b_5-\frac{1}{3}\,b_4-\frac{2}{3}\,b_1+\frac{2}{3}\,b_2 & v\cdot\lambda_2 &= \frac{1}{3}\,b_5-\frac{1}{3}\,b_4-\frac{2}{3}\,b_1-\frac{1}{3}\,b_2\\
v\cdot\lambda_3 &= -\frac{2}{3}\,b_5-\frac{1}{3}\,b_4-\frac{2}{3}\,b_1-\frac{1}{3}\,b_2 & v\cdot\lambda_4 &= \frac{1}{3}\,b_4-\frac{1}{3}\,b_1+\frac{1}{3}\,b_3+\frac{2}{3}\,b_6\\
v\cdot\lambda_5 &= -\frac{1}{3}\,b_5+\frac{1}{3}\,b_2-\frac{1}{3}\,b_3-\frac{2}{3}\,b_6 & v\cdot\lambda_6 &= \frac{1}{3}\,b_5+\frac{2}{3}\,b_4+\frac{1}{3}\,b_1+\frac{2}{3}\,b_2\\
v\cdot\lambda_7 &= \frac{1}{3}\,b_5-\frac{1}{3}\,b_4+\frac{1}{3}\,b_1+\frac{2}{3}\,b_2 & v\cdot\lambda_8 &= -\frac{2}{3}\,b_4-\frac{1}{3}\,b_1+\frac{1}{3}\,b_3+\frac{2}{3}\,b_6\\
v\cdot\lambda_9 &= \frac{1}{3}\,b_4-\frac{1}{3}\,b_1-\frac{1}{3}\,b_6+\frac{1}{3}\,b_3 & v\cdot\lambda_{10} &= -\frac{2}{3}\,b_4-\frac{1}{3}\,b_1-\frac{1}{3}\,b_6+\frac{1}{3}\,b_3\\
v\cdot\lambda_{11} &= \frac{1}{3}\,b_4-\frac{1}{3}\,b_1-\frac{2}{3}\,b_3-\frac{1}{3}\,b_6 & v\cdot\lambda_{12} &= -\frac{2}{3}\,b_4-\frac{1}{3}\,b_1-\frac{2}{3}\,b_3-\frac{1}{3}\,b_6\\
v\cdot\lambda_{13} &= \frac{2}{3}\,b_5+\frac{1}{3}\,b_2+\frac{1}{3}\,b_6+\frac{2}{3}\,b_3 & v\cdot\lambda_{14} &= \frac{2}{3}\,b_5+\frac{1}{3}\,b_2-\frac{1}{3}\,b_3+\frac{1}{3}\,b_6\\
v\cdot\lambda_{15} &= -\frac{1}{3}\,b_5+\frac{1}{3}\,b_2+\frac{1}{3}\,b_6+\frac{2}{3}\,b_3 & v\cdot\lambda_{16} &= -\frac{1}{3}\,b_5+\frac{1}{3}\,b_2-\frac{1}{3}\,b_3+\frac{1}{3}\,b_6\\
v\cdot\lambda_{17} &= \frac{2}{3}\,b_5+\frac{1}{3}\,b_2-\frac{1}{3}\,b_3-\frac{2}{3}\,b_6 & v\cdot\lambda_{18} &= -\frac{1}{3}\,b_5-\frac{2}{3}\,b_2+\frac{2}{3}\,b_3+\frac{1}{3}\,b_6\\
v\cdot\lambda_{19} &= -\frac{1}{3}\,b_5-\frac{2}{3}\,b_2-\frac{1}{3}\,b_3+\frac{1}{3}\,b_6 & v\cdot\lambda_{20} &= -\frac{1}{3}\,b_5-\frac{2}{3}\,b_2-\frac{2}{3}\,b_6-\frac{1}{3}\,b_3\\
v\cdot\lambda_{21} &= \frac{1}{3}\,b_5+\frac{2}{3}\,b_4+\frac{1}{3}\,b_1-\frac{1}{3}\,b_2 & v\cdot\lambda_{22} &= \frac{1}{3}\,b_5-\frac{1}{3}\,b_4+\frac{1}{3}\,b_1-\frac{1}{3}\,b_2\\
v\cdot\lambda_{23} &= -\frac{2}{3}\,b_5+\frac{2}{3}\,b_4+\frac{1}{3}\,b_1-\frac{1}{3}\,b_2 & v\cdot\lambda_{24} &= -\frac{2}{3}\,b_5-\frac{1}{3}\,b_4+\frac{1}{3}\,b_1-\frac{1}{3}\,b_2\\
v\cdot\lambda_{25} &= \frac{1}{3}\,b_4+\frac{2}{3}\,b_1+\frac{1}{3}\,b_3+\frac{2}{3}\,b_6 & v\cdot\lambda_{26} &= \frac{1}{3}\,b_4+\frac{2}{3}\,b_1+\frac{1}{3}\,b_3-\frac{1}{3}\,b_6\\
v\cdot\lambda_{27} &= \frac{1}{3}\,b_4+\frac{2}{3}\,b_1-\frac{2}{3}\,b_3-\frac{1}{3}\,b_6
\end{aligned}
\tag{A.5}
$$

This gives the matrix $v{\cdot}H$ in terms of the Fayet-Iliopoulos terms and thus the Casimir operators (A.3). In our normalization, the weights have length squared $\lambda\cdot\lambda = 2/3$, which corresponds to $\mathrm{Tr}\,(T_aT_b) = 3\delta_{ab}$ in the fundamental representation.

[6]In these formulas, we have eliminated b_7 using the relation $\sum_1^3 b_i + 2\sum_4^6 b_i + 3b_7 = 0$. It is straightforward to use this formula to eliminate one of b_1, b_2, b_3, to get the expressions in terms of the more standard simple roots. We have also switched the signs of b_1, b_2, b_3 as compared to the text to agree with the usual conventions for the simple roots, which do not agree with the signs we read off from the quiver diagrams.

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